DAMPING IN STRUCTURES WITH RIGID BODY DISPLACEMENT COMPONENTS

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Short Title Form:

DAMPING IN STRUCTURES

ABSTRACT

A structure attached to a rigid foundation subjected to ground motion undergoes components of rigid body motion as well as elastic deformations. Viscous damping may be defined as a function of total, elastic, or rigid body velocity. It is shown that damping proportional to elastic velocity is a form of proper damping and properly defined leads to classical modes, that damping proportional to rigid body velocity is not proper damping and inputs energy to the system, and that damping proportional to total velocity is made up of parts of the other two. Rotational components of the rigid base motion cause nonlinear input terms. Derivation of the equations of motion with transfer matrices to directly account for the base motion are given. An example is given of a framed building subjected to stochastic base motion of six components showing some of the effects of rigid body damping and rotational base motion.

INTRODUCTION

A structure subjected to ground motions, such as a building on a volume compensated mat foundation, may be modeled as an elastic assembly with discrete masses attached to a rigid foundation. Seismic waves encountering the rigid base cause its motion, which is described in translational and rotational components of a convenient coordinate center, usually the centroid. Response of each mass point of the structure consists of rigid body and elastic parts where the rigid body displacement, velocity and acceleration are functions of the base motion.

Viscous damping is defineable in several ways, but it is shown that damping proportional to the elastic velocity and the total velocity must conform to the criteria of proper damping for classical modes to exist. Damping proportional to the rigid body velocity, however, need not conform and, in fact, is not proper damping but modifies the forcing function. The forcing term is a complex function of the rigid body displacements, velocities, and accelerations transferred to each mass point. Physically, damping mechanisms relative to the base are responsible for elastic damping, but damping mechanisms relative to the fixed world contain elastic and rigid body parts. Thus, the rigid body part may arise.

A method of calculation of response is included as well as problem examples. An artificial earthquake was generated in six stochastic components of motion for displacement, velocity, and acceleration. The example problem serves to illustrate the analysis as well as some effects of the rotational components of base motion and rigid body damping.

EQUATIONS OF MOTION

The controlling equations of motion for an elastic structure modeled as a discrete system attached to a rigid base are

$$[1] \qquad [M]\{\ddot{X}_T\} + \{C\} + [K]\{X\} = \{0\}$$

where the total linear displacement $\{X_T\}_N$ is the sum of rigid body components $\{X_R\}_N$ and elastic components $\{X\}_N$

$$[2] \qquad \{X_T\}_N = \{X_R\}_N + \{X\}_N$$

Thus, each mass point has up to three orthogonal linear kinematic degrees of freedom defined for a structural total of N. The velocity and acceleration corresponding to each degree of freedom are likewise linear combinations.

$$[3a] \qquad \{\dot{X}_T\} = \{\dot{X}_R\} + \{\dot{X}\}$$

$$[3b] \qquad \{\ddot{X}_T\} = \{\ddot{X}_R\} + \{\ddot{X}\}$$

The matrix [M] is an array of the mass values at each mass point which may be defined by allocation resulting in a diagonal matrix or may contain mass coupling coefficients if determined by influence considerations. The structural stiffness is denoted by matrix [K] such that the coefficient k_{ij} is the force at i due to a unit displacement at j with all other displacements zero and i and j correspond to kinematic degrees of freedom 1 through N. The damping force vector {C} is shown as a dissipative force.

The base motion is defined at its centroid most conveniently as the vector $\{X_C\}_6$ as shown in Fig. 1 and its derivatives of velocity $\{\dot{X}_C\}_6$ and acceleration $\{\ddot{X}_C\}_6$. These vectors have six components, three linear and three rotational, which are functions of time, and in the case of earth motions, are due to seismic waves. The resulting motion $\{X_C\}$, which is nonanalytic or stochastic, may be written in two parts; $\{X_D\}_3$ for linear displacement, and $\{X_\theta\}_3$ for the rotational components.

The rigid body displacements at mass point p, which has up to 3 linear kinematic degrees of freedom, are related to the base displacement by the transfer matrix $[T_1]_p$ such that

$$[4] \qquad \underset{3}{\{X_R\}_p} = \underset{3\text{x}3}{[I]}\underset{3}{\{X_D\}} + \underset{3\text{x}3}{[T_1]_p}\underset{3}{\{X_\theta\}}$$

where

$$[5] \qquad [T_1]_p = \begin{bmatrix} 0 & r_z & -r_y \\ -r_z & 0 & r_x \\ r_y & -r_x & 0 \end{bmatrix}_p$$

The vector R_p with components r_{xp}, r_{yp} and r_{zp} locates mass point p as shown in Fig. 1 where

$$[6] \qquad R_p^2 = r_{xp}^2 + r_{yp}^2 + r_{zp}^2$$

Since the rotational components of base motion $\{X_\theta\}$ are related to the linear displacements at mass point p, successive differentiations for velocity and acceleration at p generate nonlinear terms even though small rotations are assumed such that $\sin X_{\theta i} \rightarrow 0$ and $\cos X_{\theta i} \rightarrow X_{\theta i}$. Thus, the rigid body velocity and acceleration at p are respectively

$$[7] \qquad \underset{3}{\{\dot{X}_R\}_p} = \underset{3x3}{[I]}\underset{3}{\{\dot{X}_D\}} + \underset{3x3}{[T_1]_p}\underset{3}{\{\dot{X}_\theta\}} - \underset{3x3}{[T_2]_p}\underset{3}{\{X_\theta \dot{X}_\theta\}}$$

$$[8] \qquad \underset{3}{\{\ddot{X}_R\}_p} = \underset{3x3}{[I]}\underset{3}{\{\ddot{X}_D\}} + \underset{3x3}{[T_1]_p}\underset{3}{\{\ddot{X}_\theta - X_\theta \dot{X}_\theta^2\}} - \underset{3x3}{[T_2]_p}\underset{3}{\{X_\theta \ddot{X}_\theta + \dot{X}_\theta^2\}}$$

where

$$[9] \qquad [T_2]_p = \begin{bmatrix} 0 & r_x & r_x \\ r_y & 0 & r_y \\ r_z & r_z & 0 \end{bmatrix}_p$$

$$[10a] \qquad \{X_\theta \dot{X}_\theta\}^T = [X_{\theta 1}\dot{X}_{\theta 1} \; , \; X_{\theta 2}\dot{X}_{\theta 2} \; , \; X_{\theta 3}\dot{X}_{\theta 3}]$$

$$[10b] \qquad \{\ddot{X}_\theta - X_\theta \dot{X}_\theta^2\}^T = [\ddot{X}_{\theta 1} - X_{\theta 1}\dot{X}_{\theta 1}^2 \; , \; \ddot{X}_{\theta 2} - X_{\theta 2}\dot{X}_{\theta 2}^2 \; , \; \ddot{X}_{\theta 3} - X_{\theta 3}\dot{X}_{\theta 3}^2]$$

$$[10c] \qquad \{X_\theta \ddot{X}_\theta + \dot{X}_\theta^2\}^T = [X_{\theta 1}\ddot{X}_{\theta 1} + \dot{X}_{\theta 1}^2 \; , \; X_{\theta 2}\ddot{X}_{\theta 2} + \dot{X}_{\theta 2}^2 \; , \; X_{\theta 3}\ddot{X}_{\theta 3} + \dot{X}_{\theta 3}^2]$$

Expanding Eqs. 4, 7 and 8 to include all N kinematic degrees of freedom they may be written

$$[11a] \qquad \underset{N}{\{X_R\}} = \underset{Nx3}{[\bar{I}]}\underset{3}{\{X_D\}} + \underset{Nx3}{[T_1]}\underset{3}{\{X_\theta\}}$$

$$[11b] \qquad \underset{N}{\{\dot{X}_R\}} = \underset{Nx3}{[\bar{I}]}\underset{3}{\{\dot{X}_D\}} + \underset{Nx3}{[T_1]}\underset{3}{\{\dot{X}_\theta\}} - \underset{Nx3}{[T_2]}\underset{3}{\{X_\theta \dot{X}_\theta\}}$$

$$[11c] \qquad \underset{N}{\{\ddot{X}_R\}} = \underset{Nx3}{[\bar{I}]}\underset{3}{\{\ddot{X}_D\}} + \underset{Nx3}{[T_1]}\underset{3}{\{\ddot{X}_\theta - X_\theta \dot{X}_\theta^2\}} - \underset{Nx3}{[T_2]}\underset{3}{\{X_\theta \ddot{X}_\theta + \dot{X}_\theta^2\}}$$

where the $[\bar{I}]$ are N/3 partitioned identity matrices as are the transfer matrices $[T_1]$ and $[T_2]$. Thus the rigid body motions are completely determined when all components of the base displacement, velocity, and acceleration are known.

DAMPING

Damping dissipates energy from the system thus attenuating amplitude. For analytic convenience and because of generally satisfactory results in comparison to measurement, forms of viscous damping are widely utilized in structural dynamics including earthquake engineering. Thus, the damping force in Eq. 1 may for complete generality be assumed in the form

$$[12] \qquad \{C\} = [C_E]\{\dot{X}\} + [C_R]\{\dot{X}_R\} + [C_T]\{\dot{X}_T\}$$

where the matrices $[C_i]$ are coefficients of viscous damping proportional to elastic velocities, rigid body velocities, and total velocities, respectively. Substituting Eq. 12 into Eq. 1 yields alternate form of the equation of motion

$$[13a] \quad [M]\{\ddot{X}_T\}+[C]\{\dot{X}_T\}+[K]\{X_T\}=[K]\{X_R\}+([C]-[C'])\{\dot{X}_R\}$$

$$[13b] \quad [M]\{\ddot{X}\}+[C]\{\dot{X}\}+[K]\{X\}= -[M]\{\ddot{X}_R\}-[C']\{\dot{X}_R\}$$

in which

$$[14a] \qquad [C] = [C_E] + [C_T]$$

[14b] $$[C'] = [C_R] + [C_T]$$

[14c] $$[C] - [C'] = [C_E] - [C_R]$$

In either form, the proper viscous damping term [C] is shown to be a linear combination of elastic and total damping coefficients as defined in Eq. 12.

It is also seen that damping may input energy to the system. In Eq. 13b by defining $[C_E]$ only as nonzero, that is letting [C']=[0], it can be seen that the damping does not input energy to the system as effects elastic displacements.

For classical modes to exist, it has been shown (Caughey 1960) that the damping matrix must be transformable to an orthogonal form by the same operators which uncouple the mass and stiffness matrices. Thus the matrix [C] and therefore $[C_E]$ and $[C_T]$ must be of the same form. One such sufficient form (Caughey 1960; Saul, Tantichaiboriboon and Jayachandran 1974) consists of a combination of the mass and stiffness matrices

[15] $$[C] = [M] \sum_{i=0}^{m-1} a_i [M^{-1}K]^i$$

where the coefficients a_i must be determined. If m eigenvectors $\{\phi\}_i$ where $m \leq N$ and m frequencies ω_i are determined for the undamped free vibration and written as a matrix of m vectors $[\phi]_{Nxm}$ and a diagonal matrix $\lceil \omega^2 \rfloor_{mxm}$ respectively the relationships

$$[16a] \qquad [\phi]^T[m][\phi] = \lceil M^* \rfloor$$

$$[16b] \qquad [\phi]^T[K][\phi] = \lceil M^*\omega^2 \rfloor$$

are the orthogonality relationships or uncoupling operators.

Therefore,

$$[17] \qquad [\phi]^T[C][\phi] = \lceil 2\xi\omega M^* \rfloor$$

must exist for classical modes to exist where ξ_i are the fraction of critical damping in each mode. If the fraction of critical damping is specified in only r modes where $r < m$, the remainder may be calculated from

$$[18] \qquad \xi_i = \frac{1}{2\omega_i} \sum_{j=1}^{r} a_{j-1}\omega_i^{2j-2}$$

where the coefficients a_i in Eqs. 15 and 18 are found from

[19] $$[A]_{r \times r}\{a\} = \{2\xi\omega\}$$

in which the

[20] $$A_{ij} = \omega_i^{2j-2} \qquad i,j = 1,2,\ldots,r$$

By specification of the fraction of critical damping in a limited number of modes and calculation or contrivance to get the coefficients a_i, various well known models (Saul, Tantichaiboriboon and Jayachandran 1974) for damping are obtained. Thus Rayleigh damping, a_0 and a_1 only are nonzero, consists of a linear combination of absolute and relative damping represented by viscous dampers connecting each mass to the base and to each other respectively. Taking more terms in the series of Eq. 15 results in models not as easily conceptualized.

The damping matrix $[C_R]$ is not involved in the eigenvalue problem and thus does not have to conform to the restraints on [C] for classical modes to exist.

SOLUTION OF EQUATION OF MOTION

Either of the field equations, Eq. 13a and b, may be used. Equation 13a yields the total displacement $\{X_T\}$ and so a further step is required to determine the elastic displacement from Eqs. 2 and 11a

[21] $$\{X\} = \{X_T\} - [\bar{T}]\{X_D\} - [T_1]\{X_\theta\}$$

In addition, the right hand side of Eq. 13a is of simpler form involving only displacement and velocity components of the base motion $\{X_C\}$. Equation 13b yields the elastic response directly, but the right hand side is more complex and involves all components of the base motion. Initial conditions from Eq. 13a involve rigid body motion plus elastic; Eq. 13b only elastic. The initial conditions may be specified by

[22a] $$\{X(t=0)\} = \{X_0\} \;, \quad \{X(t=0)\} = \{X_0\}$$

[22b] $$\{X_T(t=0)\} = \{X_0\} + [\bar{T}]\{X_{D0}\} + [T_1]\{X_{\theta 0}\}$$

[22c] $$\{X_T(t=0)\} = \{X_0\} + [\bar{T}]\{\dot{X}_{D0}\} + [T_1]\{\dot{X}_{\theta 0}\} - [T_2]\{X_{\theta 0}\dot{X}_{\theta 0}\}$$

In the following, Eq. 13b is solved but Eq. 13a may be easily substituted.

By assuming the displacement to be a linear combination of displacements in m modes it may be written

[23] $$\underset{N}{\{X\}} = \underset{Nxm}{[\phi]}\underset{m}{\{Z\}}$$

Substituting Eq. 23 into Eq. 13b and premultiplying by $[\phi]^T$ yields

[24] $$\ddot{Z}_i + 2\xi_i\omega_i\dot{Z}_i + \omega_i^2 Z_i = \frac{F_i}{M_i^*} f_i(t)$$

where the orthogonality relationships of Eqs. 16 and 17 uncouple the differential equation, F_i is a coefficient, $f_i(t)$ the nondimensionalized function of time, and

[25] $$\{Ff(t)\} = -[\phi]^T\left([m]\{\ddot{X}_R\}+[C']\{\dot{X}_R\}\right)$$

or if the right hand side of Eq. 13b is designated $\{F'(t)\}$ the components of $\{Ff(t)\}$ are

[26] $$F_i f_i(t) = \sum_{j=1}^{m} \phi_{ji} F'_j(t)$$

The initial conditions in the uncoupled coordinates are

[27a] $$\underset{m}{\{Z_0\}} = \underset{mxm}{\lceil M^* \rfloor^{-1}} \underset{mxn}{[\phi]^T} \underset{nxn}{[M]} \underset{n}{\{X_0\}}$$

[27b] $$\{\dot{Z}_0\} = \lceil M^* \rfloor^{-1} [\phi]^T [M] \{\dot{X}_0\}$$

and the solution may be written

[28] $$Z_i(t) = e^{-\xi_i \omega_i t} \left[\frac{\dot{Z}_{0i} + \xi_i \omega_i Z_{0i}}{\omega_{di}} \sin\omega_{di} t + Z_{0i} \cos\omega_{di} t \right]$$

$$+ \frac{F_i}{M_i^* \omega_{di}} \int_0^t f_i(t') e^{-\xi_i \omega_i (t-t')} \sin\omega_{di}(t-t') dt'$$

where

[29] $$\omega_{di} = \omega_i \sqrt{1 - \xi_i^2}$$

If it is assumed that $f_i(t)$ is linear over short intervals of time, i.e., piecewise linear, a closed form of solution may be written (Saul, Tantichaiboriboon and Jayachandran 1974) for $Z_i(t)$. Approximating $f(t)$ as a quadratic form enables increased accuracy in the numerical solution.

If the solution of the equations of motion by numerical integration is preferred, the damping matrices may be directly calculated once the ξ_i are determined from

$$[C] = [M]\ [\phi] \left\lceil \frac{2\xi\omega}{M^*} \right\rfloor [\phi]^T [M] \qquad [30]$$

nxn nxn nxm mxn mxn nxn

EXAMPLE PROBLEM

The six correlated components of base motion for displacement, velocity and acceleration were generated by simulating an artificial earthquake from an acceleration energy spectrum $S^a[\omega]$ (Housner and Jennings 1964)

$$[31] \qquad S^a[\omega] = \frac{c\left(1 + \frac{\omega^2}{147.8}\right)}{\left(1+\frac{\omega^2}{242}\right)^2 + \left(\frac{\omega^2}{147.8}\right)}$$

where c = 0.01238 ft^2/sec^3(11.5 cm^2/sec^3), by using fast Fourier transforms. Weak correlation was achieved using statistical data from measured translational ground motion (Chen 1975) of earthquakes. The artificial earthquake was treated as a multi-dimensional nonstationary stochastic process by specifying a deterministic envelope function conforming to recorded earthquakes (Jennings, et. al. 1968).

A 2-story single bay frame building on a cellular foundation was dynamically analyzed as an eight degree of freedom system due to rigid body motion of the base. The building, Fig. 2., is regular and has 2 axes of symmetry. Sidesway degrees of freedom only were chosen, since there is a discernable gap in the natural frequencies above the first 8 (1.26 Hz - 8.14 Hz). The stiffness and mass matrices are given in Table 1 along with the eigenvectors and frequencies. The center of gravity of the cellular foundation is not coincident with the center of symmetry of the structure, as seen in Fig. 2. The structure was subjected to a 10 sec. artificial earthquake with ξ_1=0, 0.75, & 1.5 and ξ_2 = 0, 0.375, & 0.75 respectively being specified for proper (elastic) damping. Rigid body damping was varied to check its effect on the forcing function and on the response of the structure. In addition, the rotational components of the base motion were suppressed to ascertain their effect on the response and on the forcing function. A typical response curve is given in Fig. 3 and Fig. 4 shows a sample of the forcing function $\{F'(t)\}$.

CONCLUSIONS

The equations of motion for a structure attached to a rigid foundation undergoing linear $\{X_D\}$ and rotational $\{X_\theta\}$ movements due to earthquake have a forcing function, from Eqs. 13b, 11b and 11c, containing nonlinear terms as well as a form of damping

$$[32]\ \{F'(t)\} = -[M]\left([\bar{I}]\{\ddot{X}_D\} + [T_1]\{\ddot{X}_\theta - X_\theta \dot{X}_\theta^2\} - [T_2]\{X_\theta \ddot{X}_\theta + \dot{X}_\theta^2\}\right)$$

$$-[C']\left([\bar{I}]\{\dot{X}_D\} + [T_1]\{\dot{X}_\theta\} - [T_2]\{X_\theta \dot{X}_\theta\}\right)$$

The $X_{\theta i}$ terms are not necessarily insignificant. Although usually smaller in magnitude than the linear terms and reduced further as cross products or squares when multiplied by a transfer matrix they may become comparable. The damping [C'] results only when the damping mechanism is conceptualized as being attached to the fixed world. When [C'] is of significant magnitude, it can provide energy to {F'(t)} comparable to the mass coupling term. If all damping is modeled as being relative to the base only proper damping results and [C'] becomes zero. For classical modes to exist, the form of proper damping is given in Eq. 17. Although the fractions of critical damping may be assigned to a limited number of modes and calculated for the remainder assigning values of ξ_i for all modes, either equal or decreasing in higher modes appears to be as reasonable.

Numerical solution of the equations of motion by superposition of modes or direct integration is possible. Either way determination of a damping matrix may be needed as given by Eq. 30. Superposition of modes enables limiting the calculation to significantly contributing modes only.

The example problem showed that small changes of rigid body damping had slight effect on response and that elastic damping, which is to be expected, had much greater effect.

Both rigid body and elastic damping could be combined to yield total damping. The effects of rotational components of the base motion must be considered with rigid body damping since it is their product which can add energy to the system.

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Table 1 Properties of Example Structure

Property	i	1	2	3	4	5	6	7	8
	1	5.325	-0.847	-1.288	1.288	-4.496	-0.001	-0.018	0.018
	2		5.325	1.288	-1.288	-0.001	-4.496	0.018	-0.018
	3			40.923	-1.951	-0.011	0.011	-41.290	-0.018
[K]	4				40.922	0.011	-0.011	-0.018	-41.290
(kips/in)	5					11.881	-0.874	-1.294	1.294
	6						11.881	1.294	-1.294
	7							73.553	-1.937
	8								73.551
[M] ($k\text{-}s^2/in$)		0.0317	0.0317	0.0317	0.0317	0.0453	0.0453	0.0453	0.0453

Table 1 (cont.)

Period t_i (s)		0.795	0.643	0.362	0.351	0.330	0.304	0.125	0.123
Frequency f_i (Hz)		1.258	1.555	2.761	2.851	3.030	3.285	7.974	8.137
	1	-3.181	-3.319	0.888	-2.204	2.181	0.090	0.000	0.000
	2	3.181	-3.319	0.888	2.204	2.181	-0.090	0.000	0.000
	3	-0.695	0.000	-2.908	-0.274	-0.000	-2.600	2.574	-3.023
[a]	4	0.696	0.000	2.907	0.274	-0.000	2.599	2.575	-3.024
	5	-1.843	-1.824	+0.239	+2.753	-2.776	-0.065	0.000	0.000
	6	-1.843	-1.824	-0.239	-2.753	-2.776	0.065	0.000	0.000
	7	-0.472	0.000	-2.124	-0.072	0.000	2.510	2.529	-2.154
	8	0.472	0.000	2.124	0.072	-0.000	2.509	-2.530	-2.154

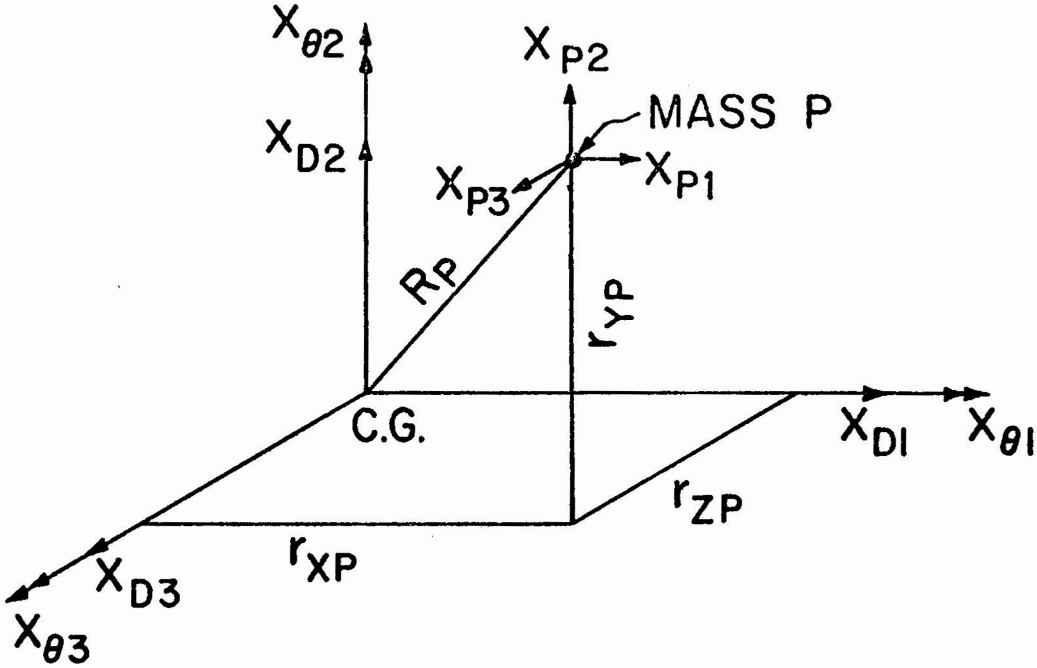

Fig.1 Components of Motion Defined at Centroid of Base Cand at a Typical Mass Point p.

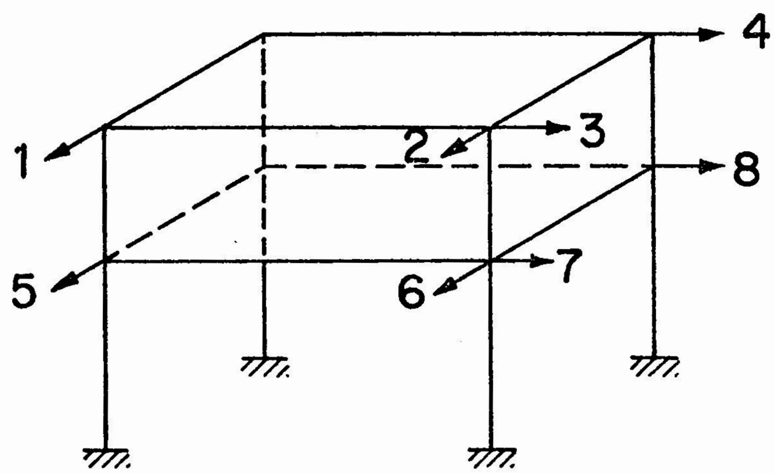

Fig. 2a Dynamic Degrees of Freedom of Frame Used in Example Problem

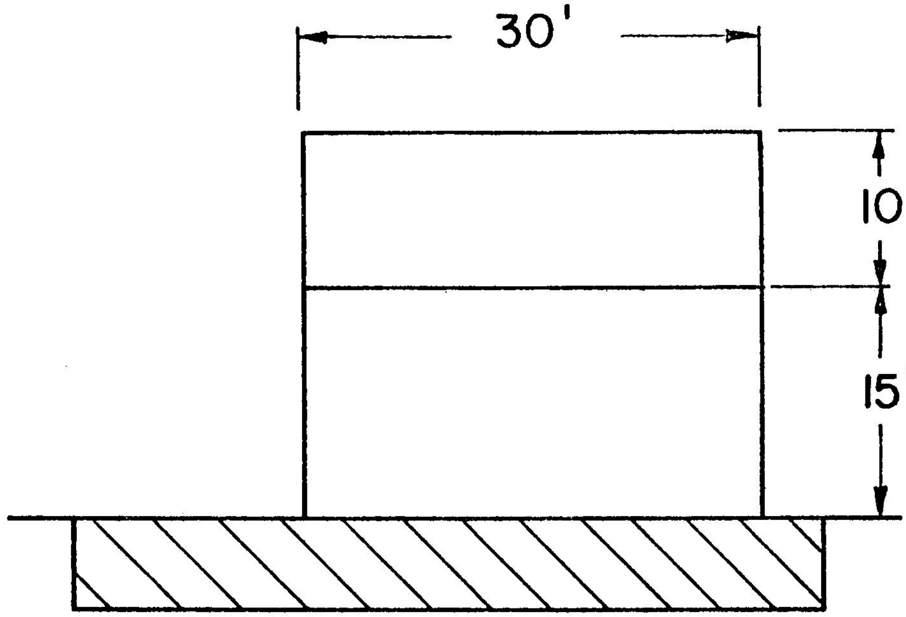

Fig. 2b Elevation of 2-Story Building Frame Showing Rigid Base

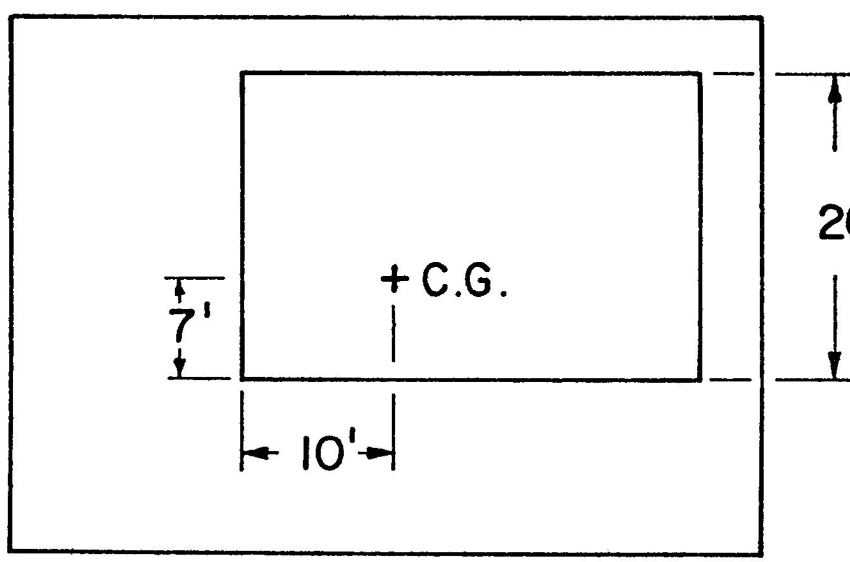

Fig. 2c Plan of Structure in Example Problem

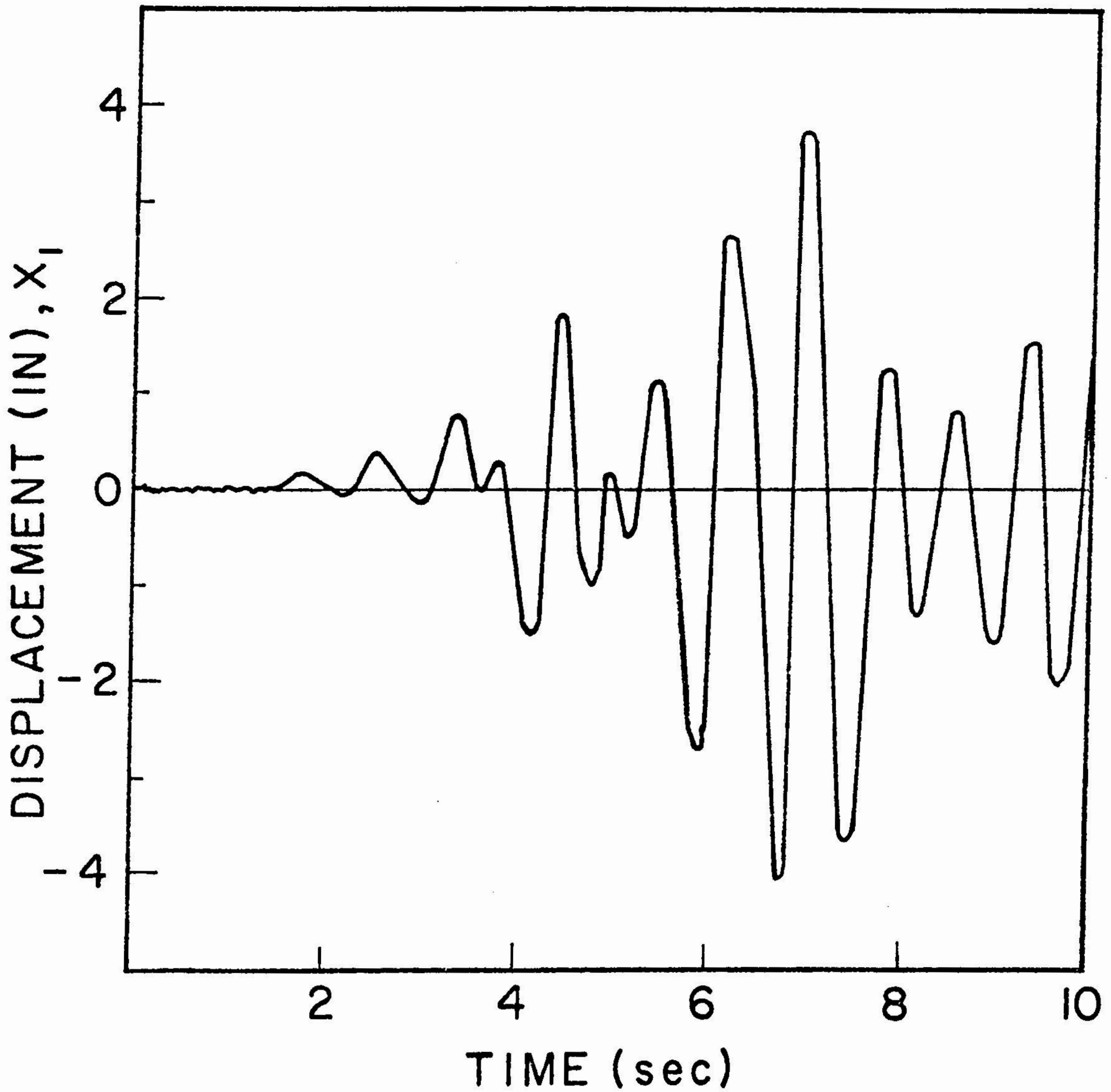

Fig. 3 Typical Displacement Response of Structure due to Artificial Earthquake

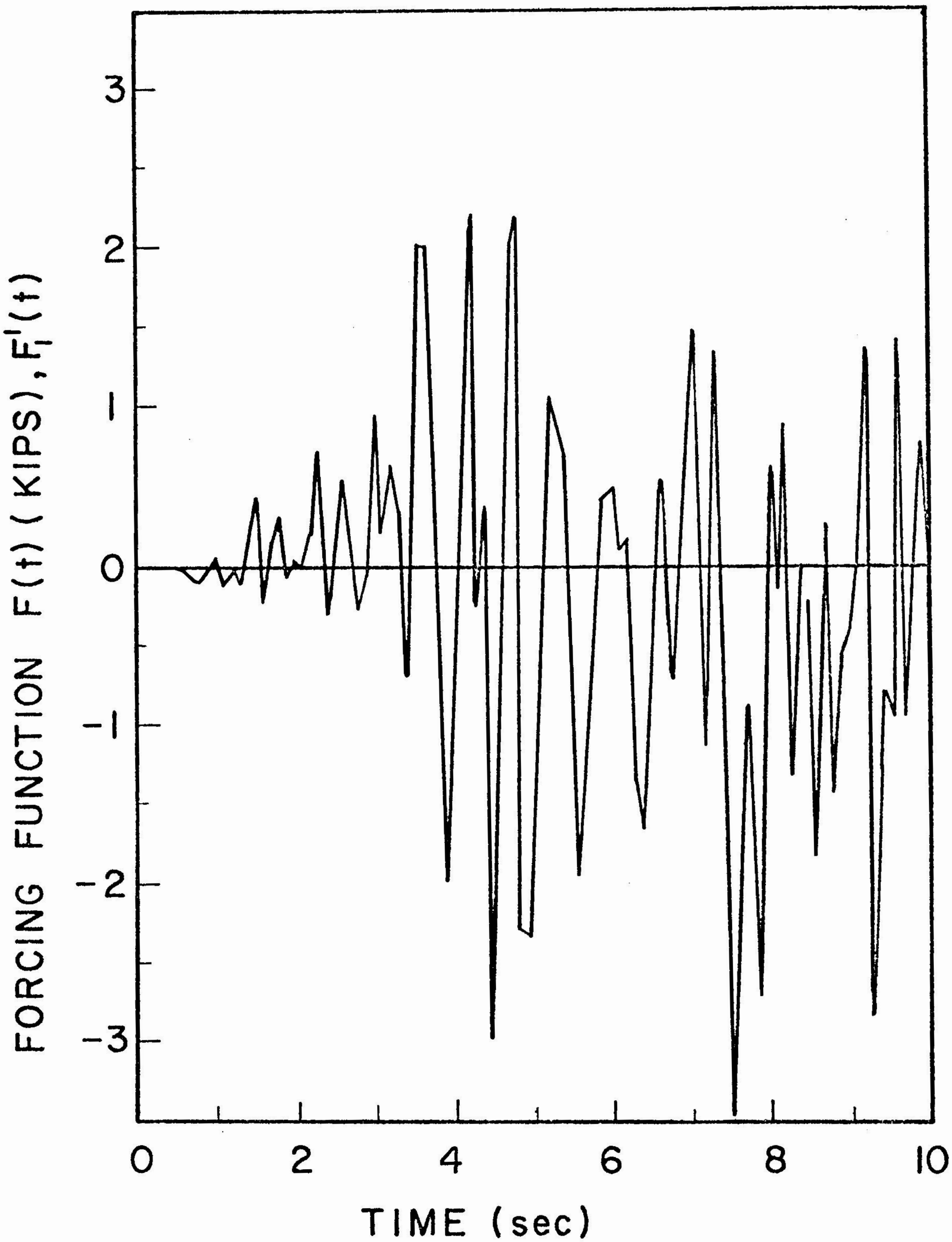

Fig. 4 Artificial Earthquake Generated Forcing Function Component